[1963] 39 p. ref

THE OPTIMIZATION OF MHD GENERATORS WITH ARBITRARY CONDUCTIVITY*

N64 13400 *
code 1

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(NASA CR 55183)

UNPUBLISHED PRELIMINARY DATA

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ABSTRACT

This analysis is based on the one-dimensional, inviscid, non-heat-conducting flow equations of an ionized gas (whose electrical conductivity is in general a function of pressure and temperature) flowing through a channel for the purpose of the extraction of electrical power. The problem is: given the inlet conditions and a fixed channel length, what should be the distribution of channel cross-sectional area (and hence of all other gas properties) in order to extract maximum power? This variational problem is solved in the present paper by means of a computational procedure based on the "method of gradients". The method developed here can be applied to either a continuous-electrode generator or a segmented-electrode generator, and with tensor conductivity.

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*The analytical portion of this paper was supported by the Space Science Laboratory of the General Electric Company. The computation and final completion of this work was supported by the National Aeronautics and Space Administration under ~~contract~~ (NASA Grant NsG-316) to the Institute for Direct Energy Conversion of the Towne School of Civil and Mechanical Engineering of the University of Pennsylvania.

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Two series of calculations were performed. In the first series, the conductivity was assumed to vary with T^{ω}. Quick convergence to an optimum distribution was obtained with $\omega = 10$ for all inlet Mach numbers used except $M_o^2 = 2$. The optimum powers extracted were then compared for various inlet Mach numbers, both for constant inlet temperature and for constant inlet stagnation temperature. In the second series, conductivity was assumed to vary with $T^{10}/\sqrt{p}$. It was found that the power extracted keeps increasing as exit pressure decreases and no maximum power exists for finite exit area. With practical limits for exit-to-inlet area ratios of 10 and 20, the optimum extracted power was then obtained for various inlet Mach numbers. As expected, the improvement over the constant velocity distribution was great.

AUTHOR

Nomenclature

The following nomenclature is used in the paper:

A	=	cross-sectional area of MHD channel
A_o	=	cross-sectional area at inlet of MHD channel
A_1	=	cross-sectional area at exit of MHD channel
B	=	magnetic field strength
C	=	$(1 - K) G_o$
C_p	=	specific heat of constant pressure
f_i	=	functions of α, x, and y_i; $i = 1, 2, \ldots, n$
G	=	constant
K	=	generator load factor
L	=	length of MHD channel
M	=	Mach number
M_o	=	Mach number at inlet of MHD channel
p	=	gas pressure
p_o	=	gas pressure at inlet of MHD channel
p_1	=	gas pressure at exit of MHD channel
G_o	=	magnetic interaction parameter
R	=	gas constant
T	=	gas temperature
T_o	=	gas temperature at inlet of MHD channel
T_1	=	gas temperature at exit of MHD channel
u	=	gas velocity
u_o	=	gas velocity at inlet of MHD channel
u_1	=	gas velocity at exit of MHD channel
x	=	distance along MHD channel from the inlet of the channel
y_i	=	dependent variables; $i = 1, 2, \ldots, n$
y_{iL}	=	values of y_i at $x = L$
α	=	driving function
α_o	=	assigned value of $\alpha(x)$ in the first of successive computations

γ	=	specific heat ratio of a perfect gas
λ_i	=	influence functions; $i = 1, 2, \ldots, n$
ρ	=	gas density
σ	=	gas conductivity
ϕ_o	=	enthalpy at inlet of MHD channel
ϕ_1	=	optimum enthalpy at exit of MHD channel
$\bar{\phi}_1$	=	enthalpy at exit of MHD channel obtained from first calculation in the iterative procedure
ω_o	=	cyclatron frequency
τ	=	collision time

1. INTRODUCTION

In several previous reports (1, 2), Sutton has investigated the one-dimensional MHD flow for power generation. The five flow configurations investigated were: constant velocity, constant area, constant temperature, constant pressure, and constant density. Calculations in Ref. 1 showed that for a given channel length, the constant velocity distribution yields the largest amount of power among the five cases investigated. Although the constant-velocity distribution is probably not too far from the optimum, it is by no means clear that it is the true optimum among the infinitely many possible distributions. One would intuitively suspect that, since the optimum distribution must be dependent on the particular function which governs the variation of conductivity, the constant-velocity distribition which ignores this dependency is unlikely to be the true optimum.

To seek a true optimum velocity distribution (and hence all the other variables, including cross-sectional area), one might at first be tempted to use the classical approach of calculus of variations, i.e., the Euler-Lagrange equations with Lagrange multipliers. This was indeed tried. It turns out that in this problem, the end points (corresponding to the properties at the two ends of the channel) are not all fixed. Neither are the so-called "natural boundary conditions" satisfied. In fact, the end-point at the upper integration limit is precisely one of the quantities that we wish to maximize. Hence the usual variational method fails in this problem. An attempt to transform the independent variable from distance x to stagnation temperature T_{stag} simplifies the calculation greatly (due to the simultaneous reduction of one dependent variable and one constraint equation), but unfortunately does not seem to remove the essential difficulty. Furthermore, it can be shown that for a class of problems of which the present problem is a special case, the classical Euler-Lagrange formulation always leads to a singular solution. A different approach is therefore used in this paper. This is the method of gradients (also know as the method of steepest descent) and is to be briefly explained in the next section.

The analytical portion of this paper was first issued as a report of limited circulation by the Space Science Laboratory of the General Electric Company (3).

2. THE METHOD OF GRADIENTS

The application of the method of gradients to a variational problem was apparently first proposed by Courant in 1941 (4). Recently it has been applied by Kelly (5) and Bryson (6) and their co-workers to the optimization of flight trajectories in satellite and space vehicle re-entry. The main concept can be summarized as follows:

Consider a set of (n + 1) functions

$$y_1, \quad y_2, \quad y_3, \quad \cdot \cdot \cdot, y_n, \ \alpha$$

which are all functions of the independent variable x, between $x = 0$ and $x = L$. For convenience, we seek out one of them, α, and call it the "driving function". (In most problems, one of the functions is the dominant variable of the problem and therefore is the obvious choice as the driving function. In other problems, however, the choice may not be clear-cut; in such cases the particular selection turns out to be unimportant.) The remaining n functions can be considered as the "dependent variables". They are governed by the n known equations:

$$\frac{dy_i}{dx} = f_i(y_1, y_2, \cdots, y_n, \alpha, x)\,; \qquad i = 1, 2, \cdots, n$$

It is to be noted that, including α, we have n + 1 unknown functions but only n equations. Hence one of the functions, say $\alpha(x)$, can be arbitrarily assigned. Now we wish to modify $\alpha(x)$ in order to maximize (or minimize) a certain quantity ϕ which is a function of the final values $y_{1L}, y_{2L}, \cdots, y_{nL}$ of the dependent variables $y_1, y_2, \cdots, y_n$. That is, we wish to find the particular $\alpha(x)$ such that

$$\phi = \phi(y_{1L}, y_{2L}, \cdots, y_{nL})$$

takes a maximum (or minimum) value.

First of all, we seek the effect of small perturbations around an initial solution (i.e., the zero-th order approximation) and write:

$$\frac{d}{dx}(\delta y_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\delta y_j + \frac{\partial f_i}{\partial \alpha}\delta\alpha \tag{1}$$

Next we define a set of "influence functions" λ_i such that:

$$\frac{d\lambda_i}{dx} = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial y_i}\lambda_j \tag{2}$$

Multiplying Eq. (1) by λ_i and Eq. (2) by δy_i and summing over i, we obtain, noting that the resulting two quantities with double summations cancel out,

$$\frac{d}{dx}\sum_{i=1}^{n}\lambda_i\delta y_i = \sum_{i=1}^{n}\lambda_i\frac{\partial f_i}{\partial \alpha}\delta\alpha$$

Integrating over the distance x, from $x = 0$ to $x = L$, we obtain:

$$\left[\sum_{i=1}^{n} \lambda_i \delta y_i\right]_0^L = \int_0^L \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \alpha} \delta\alpha(x)\, dx \tag{3}$$

To solve for λ_i by Eqs. (2), we must assign boundary values to λ_i. Here we assign these values at $x = L$ such that:

$$\lambda_{iL} = \left[\frac{\partial \phi}{\partial y_i}\right]_{x=L} \tag{4}$$

With this choice, the left side of Eq. (3) becomes:

$$\left[\sum_{i=1}^{n} \lambda_i \delta y_i\right]_0^L = \delta\phi - \left[\sum_{i=1}^{n} \lambda_i \delta y_i\right]_{x=0} \tag{5}$$

In the problem that we shall consider, the values of the dependent variables at $x = 0$ are fixed and therefore the last term on the right side of the above equation can be dropped. Substituting into the left side of Eq. (3):

$$\delta\phi = \int_0^L \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \alpha} \delta\alpha(x)\, dx \tag{6}$$

This equation enables us to calculate, for a small perturbation $\delta\alpha(x)$ of the driving function $\alpha(x)$, the change $\delta\phi$ on the function ϕ which we wish to optimize. (Note that $\lambda_i(x)$ has already been obtained from the differential equations (2), together with the boundary conditions (4)). Now, for a given value of $\int_0^L (\delta\alpha)^2 dx$, it can be shown by means of Lagrange multipliers, that the largest value of $\delta\phi$ is obtained if

$$\delta\alpha = G\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha} ; \qquad\qquad G = \text{constant} \tag{7}$$

This represents the "s t e e p e s t d e s c e n t" direction towards the minimum ϕ. For an $\delta\alpha$ along this direction,

$$\delta\phi = G\int_0^L \Big[\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\Big]^2 dx \tag{8}$$

The constant G is determined by the size of step, i.e., the value of $\delta\phi$, that we wish to take in each cycle of calculation. For example, if we wish to take $\delta\phi$ to be –1% of ϕ_1, we will then use:

$$G = \frac{\delta\phi}{\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx} ; \qquad\qquad \delta\phi = -0.01\phi \tag{9}$$

The new $\alpha(x)$ is then

$$\alpha(x)_{new} = \alpha(x)_{old} + G\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha} \tag{10}$$

and the calculation is repeated. It is clear that with a different $\alpha(x)$, all the $y_i(x)$ values will be different also. In this way each cycle of calculation yields a modification of $\alpha(x)$, that is the $\delta\alpha(x)$ which will bring ϕ closer to its optimum value. The calculation can be terminated when $\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx$ is much smaller than its value during the first calculation.

3. MHD PROBLEM

We shall now apply the above discussion to the MHD channel flow problem. With the usual assumptions of one-dimensional approximation, perfect gas law, and non-viscous and non-heat-conducting fluid, (see Ref. 1), the relevant equations are

Momentum $\rho u \frac{du}{dx} + \frac{dp}{dx} = -(1-K)B^2 \sigma u$

Energy $\rho u \frac{d}{dx}(C_p T + \frac{u^2}{2}) = -K(1-K) B^2 \sigma u^2$

Continuity $\rho A u = constant$

Perfect Gas $p = \rho R T$

In the above equations, σ is the conductivity and K is the loading factor; i.e., the ratio of actual voltage to open-circuit voltage. The above equations apply to either a continuous-electrode generator with $\omega_o \tau \ll 1$ or a segmented-electrode generator with arbitrary $\omega_o \tau$, where ω_o is the cyclotron frequency and τ the collision time. However, they can also be applied to a continuous-electrode generator with arbitrary $\omega_o \tau$, if σ is replaced by $\frac{\sigma_s}{1+\omega_o^2 \tau^2}$, where σ_s is the conductivity with B = 0.

Given the initial conditions at x = 0, we wish to minimize the <u>final</u> stagnation enthalpy $(C_p T + \frac{u^2}{2})_L$ at x = L.

Let $\frac{du}{dx} = \alpha(x)$ be the driving function. The dependent variables are u, p and T. Let

$$y_1 = u, \qquad y_2 = p, \qquad y_3 = T$$

The set of equations $\frac{dy_i}{dx} = f_i(y_1, y_2, \cdots, y_n, \alpha, x)$ are therefore:

$$\frac{du}{dx} = \alpha \tag{11a}$$

$$\frac{dp}{dx} = -(1-K)B^2\sigma u - \frac{p}{RT}u\alpha \tag{11b}$$

$$\frac{dT}{dx} = -\frac{K(1-K)B^2R}{C_p}\left(\frac{\sigma u T}{p}\right) - \frac{u}{C_p}\alpha \tag{11c}$$

with $\sigma = \sigma(p, T)$

Before proceeding any further, it is convenient to non-dimensionalize every quantity. For this purpose we define:

$$\bar{u} = \frac{u}{u_o} \qquad \bar{p} = \frac{p}{p_o} \qquad \bar{T} = \frac{T}{T_o} \qquad \bar{\rho} = \frac{\rho}{\rho_o}$$
$$\bar{x} = \frac{x}{L} \qquad \bar{\alpha} = \frac{d\bar{u}}{d\bar{x}} \qquad \bar{\sigma} = \frac{\sigma}{\sigma_o} \tag{12}$$

Furthermore, we introduce the magnetic interaction parameter Q_o and Mach number M_o

$$Q_o = \frac{B^2\sigma_o L}{\rho_o u_o} \quad ; \quad M_o = \frac{u_o}{\sqrt{\gamma R T_o}} \tag{13}$$

(Note that the subscript 0 denotes initial conditions at $x = 0$). Having defined these quantities in non-dimensional form, we shall in the following drop the bar on top, with the understanding that <u>all</u> quantities are now non-dimensional in the manner just defined.

The equations for the non-dimensional dependent variables u, p, and T are:

$$\frac{du}{dx} = \alpha \tag{14a}$$

$$\frac{dp}{dx} = -(1-K)\gamma Q_0 M_0^2 u\sigma - \gamma M_0^2 \frac{pu\alpha}{T} \tag{14b}$$

$$\frac{dT}{dx} = -K(1-K)(\gamma-1)Q_0 M_0^2 \frac{\sigma u T}{p} - (\gamma-1)M_0^2 u\alpha \tag{14c}$$

Let

$$C \equiv (1-K)Q_0 \tag{15}$$

The above equations can be rewritten:

$$\frac{du}{dx} = \alpha \tag{16a}$$

$$\frac{dp}{dx} = -\gamma M_0^2 \left(Cu\sigma + \frac{pu\alpha}{T}\right) \tag{16b}$$

$$\frac{dT}{dx} = -(\gamma-1)M_0^2 \left(KC \frac{\sigma u T}{p} + u\alpha\right) \tag{16c}$$

The equations for the influence functions λ_u, λ_p, and λ_T are

$$\frac{d\lambda_u}{dx} = M_0^2\left\{\gamma\left(C\sigma + \frac{p\alpha}{T}\right)\lambda_p + (\gamma-1)\left(KC\frac{\sigma T}{p} + \alpha\right)\lambda_T\right\} \tag{17a}$$

$$\frac{d\lambda_p}{dx} = M_0^2\left\{\gamma\left(Cu\frac{\partial\sigma}{\partial p} + \frac{u\alpha}{T}\right)\lambda_p + (\gamma-1)KCuT\frac{\partial}{\partial p}\left(\frac{\sigma}{p}\right)\lambda_T\right\} \tag{17b}$$

$$\frac{d\lambda_T}{dx} = M_0^2\left\{\gamma\left(Cu\frac{\partial\sigma}{\partial T} - \frac{pu\alpha}{T^2}\right)\lambda_p + (\gamma-1)KC\frac{u}{p}\frac{\partial}{\partial T}(\sigma T)\lambda_T\right\} \tag{17c}$$

The non-dimensional final enthalpy which is the quantity that we wish to minimize is:

$$\phi_1 = T_1 + \frac{\gamma-1}{2} M_0^2 u_1^2$$

The boundary values of $\lambda_u, \lambda_p,$ and λ_T are therefore:

$$[\lambda_u]_1 = \frac{\partial \phi}{\partial u_1} = (\gamma-1) M_0^2 u_1 \tag{18a}$$

$$[\lambda_p]_1 = \frac{\partial \phi}{\partial p_1} = 0 \tag{18b}$$

$$[\lambda_T]_1 = \frac{\partial \phi}{\partial T_1} = 1 \tag{18c}$$

The quantity proportional to our desired $\delta\alpha(x)$ is:

$$\sum_{i=1}^{3} \lambda_i \frac{\partial f_i}{\partial \alpha} = (\gamma-1) M_0^2 \left\{ \frac{\lambda_u}{(\gamma-1) M_0^2} - \frac{\gamma}{\gamma-1} \frac{pu}{T} \lambda_p - u \lambda_T \right\} \tag{19}$$

We now introduce for convenience two new quantities λ_u' and λ_p' to replace λ_u and λ_p :

$$\lambda_u' = \frac{\lambda_u}{(\gamma-1) M_0^2} \quad ; \quad \lambda_p' = \left(\frac{\gamma}{\gamma-1}\right) \lambda_p \tag{20}$$

Eqs. (17) are then simplified:

$$\frac{d\lambda_u'}{dx} = \left(c\sigma + \frac{p\alpha}{T}\right) \lambda_p' + \left(Kc \frac{\sigma T}{p} + \alpha\right) \lambda_T \tag{21a}$$

$$\frac{d\lambda_p'}{dx} = \gamma M_0^2 \left\{ \left(cu \frac{\partial \sigma}{\partial p} + \frac{u\alpha}{T}\right) \lambda_p' + KcuT \frac{\partial}{\partial p}\left(\frac{\sigma}{p}\right) \lambda_T \right\} \tag{21b}$$

$$\frac{d\lambda_T}{dx} = (\gamma-1) M_0^2 \left\{ \left(cu \frac{\partial \sigma}{\partial T} - \frac{pu\alpha}{T^2}\right) \lambda_p' + Kc \frac{u}{p} \frac{\partial}{\partial T} (\sigma T) \lambda_T \right\} \tag{21c}$$

The greatest simplication is, however, in Eqs. (18) and (19). Eqs. (18) become:

$$[\lambda_u']_1 = u_1 \tag{22a}$$

$$[\lambda_p']_1 = 0 \tag{22b}$$

$$[\lambda_T]_1 = 1 \tag{22c}$$

and Eq. (19) becomes:

$$\sum_{i=1}^{3} \lambda_i \frac{\partial f_i}{\partial \alpha} = (\gamma-1) M_o^2 (\lambda_u' - \frac{pu}{T} \lambda_p' - u\lambda_T) \tag{23}$$

It should be noted that in an MHD problem, the physical reality calls for positive and non-vanishing values of p, T, and u. Under these restrictions, it may be that a stationary value of ϕ_1 does not exist. In other words, it is possible that the optimum ϕ_1 requires zero p and T (hence infinite area) at exit, a requirement which is hard to meet in practice. This is reflected in some of the calculations. Thus for certain assumed forms of conductivity in relation to temperature or temperature and pressure, this method results in no minimum ϕ_1 despite the fact that successive computations yield progressively smaller ϕ_1. For these instances, there is no stationary value of ϕ_1 with respect to the driving function $\alpha(x)$ within the restrictions of physical reality.

It also should be noted that peculiar to the MHD problem, $\delta\alpha$ will be zero whenever $\sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \alpha}$ becomes zero, or equivalently, the right hand side of Eq. (23) vanishes. Substituting the boundary conditons given in Eqs. (22a), (22b), and (22c) into Eq. (23), one obtains $(\delta\alpha)_{x=1} = 0$ regardless of the value of G. Since $\alpha(x)$ is arbitrarily assigned in the first iteration, the value of α at x = 1 will remain this arbitrarily assigned value. It means that this method does not alter the rate of change of velocity at the exit from its initially assigned value. This appears to be purely coincidental due to the particular values of boundary conditions associated with this problem. However, we shall see later that actual computations seem to indicate that this does not pose an intrinsic dilemma to the minimization of ϕ_1.

4. SUMMARY OF COMPUTATIONAL PROCEDURE

The computational procedure can now be summarized as follows: Given the following quantities which are constants of the problem:

Generator Coefficient	K
Magnetic Parameter	Q_o
Inlet Mach Number	M_o
Ratio of Specific Heats	γ

hence, $C = (1 - K) Q_o$

and the variation of conductivity $\sigma = \sigma(p, T)$

Step 1 Assuming a distribution for the "driving function" $\alpha(x)$, we can numerically integrate the following set of simultaneous differential equations:

$$\frac{du}{dx} = \alpha \tag{16a}$$

$$\frac{dp}{dx} = -\gamma M_o^2 \left(C u \sigma + \frac{p u \alpha}{T} \right) \tag{16b}$$

$$\frac{dT}{dx} = -(\gamma - 1) M_o^2 \left(K C \frac{\sigma u T}{p} + u \alpha \right) \tag{16c}$$

The integration is to be performed from $x = 0$ to $x = 1$. The boundary conditions at $x = 0$ are $u_o = p_o = T_o = 1$. When the integration is completed and the final values u_1, p_1 and T_1 at $x = 1$ are found, we obtain $\phi_1 = T_1 + \frac{\gamma - 1}{2} M_o^2 u_1^2$

Since this value of ϕ_1 is not necessarily the minimum, we can proceed to the next step:

Step 2 Numerically integrate the following set of simultaneous differential equations for the "influence functions" λ'_u, λ'_p, and λ_T:

$$\frac{d\lambda'_u}{dx} = \left(c\sigma + \frac{p\alpha}{T}\right)\lambda'_p + \left(Kc\frac{\sigma T}{p} + \alpha\right)\lambda_T \tag{21a}$$

$$\frac{d\lambda'_p}{dx} = \gamma M_o^2\left\{\left(cu\frac{\partial\sigma}{\partial p} + \frac{u\alpha}{T}\right)\lambda'_p + KcuT\frac{\partial}{\partial p}\left(\frac{\sigma}{p}\right)\lambda_T\right\} \tag{21b}$$

$$\frac{d\lambda_T}{dx} = (\gamma-1)M_o^2\left\{\left(cu\frac{\partial\sigma}{\partial T} - \frac{pu\alpha}{T^2}\right)\lambda'_p + Kc\frac{u}{p}\frac{\partial}{\partial T}(\sigma T)\lambda_T\right\} \tag{21c}$$

This integration is to be performed "backwards" from x = 1 to x = 0, starting with the boundary conditions at x = 1:

$$[\lambda'_u]_1 = u_1 \tag{22a}$$

$$[\lambda'_p]_1 = 0 \tag{22b}$$

$$[\lambda_T]_1 = 1 \tag{22c}$$

Note that all quantities such as $u, p, T,$ and σ appearing in the coefficients of Eqs. (21) are the results of Step 1.

Step 3 The desired variation $\delta\alpha(x)$ on the driving function $\alpha(x)$ is then

$$\delta\alpha(x) = G(\gamma-1)M_o^2\left(\lambda'_u - \frac{pu}{T}\lambda'_p - u\lambda_T\right)$$

or

$$\delta\alpha(x) = G\Lambda \tag{23a}$$

where $$\Lambda = \Lambda(x) = (\gamma-1)M_o^2\left(\lambda'_u - \frac{pu}{T}\lambda'_p - u\lambda_T\right) \tag{23b}$$

The constant G is obtained from

$$G = \frac{\delta\phi}{\int_0^1 \Lambda^2 dx}$$

<u>Step 4</u> Repeat Step 1 to Step 3 with the new $\alpha(x)$

$$\alpha(x)_{new} = \alpha(x)_{old} + \delta\alpha(x) \tag{24}$$

The iteration can be terminated when $\int_0^1 \Lambda^2 dx$ is much smaller than its value in the first calculation.

5. <u>NUMBERICAL RESULTS</u>

I. Conductivity varies with T^{ω}

If conductivity is assumed to be a function of temperature only and is of the form of the power law

$$\sigma = T^{\omega} \tag{25}$$

then Eqs. (16a), (16b), and (16c) reduce to

$$\frac{du}{dx} = \alpha \qquad (16a')$$

$$\frac{dp}{dx} = -\gamma M_o^2 \left(C u T^{\omega} + \frac{p u \alpha}{T}\right) \qquad (16b')$$

$$\frac{dT}{dx} = -(\gamma - 1) M_o^2 \left(K C u \frac{T^{\omega+1}}{p} + u\alpha\right) \qquad (16c')$$

Similarily, Eqs. (21a), (21b), and (21c) reduce to

$$\frac{d\lambda_u'}{dx} = \left(C T^{\omega} + \frac{p\alpha}{T}\right)\lambda_p' + \left(K C \frac{T^{\omega+1}}{p} + \alpha\right)\lambda_T \qquad (21a')$$

$$\frac{d\lambda_p'}{dx} = \gamma M_o^2 \left(\frac{u\alpha}{T}\lambda_p' - K C u \frac{T^{\omega+1}}{p^2}\lambda_T\right) \qquad (21b')$$

$$\frac{d\lambda_T}{dx} = (\gamma-1)M_c^2\left\{\left(Cu\omega T^{\omega-1} - \frac{pu\alpha}{T^2}\right)\lambda'_p + \frac{(\omega+1)KCu}{p}T^{\omega}\lambda_T\right\} \quad (21c')$$

The above two sets of equations are to be integrated numerically by the procedure outlined above, together with the given boundary conditions.

The numberical integration was carried out with an RPC 4000 electronic digital computer. In general, the program was written in such a way that successive "descending" steps are progressively smaller, and are proportional to $\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial x}\right)^2 dx$. This is accomplished by taking $\delta\phi$ in the (n + 1)-th computation as

$$\left[\delta\phi\right]_{n+1} = \frac{\left[\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx\right]_{n+1}}{\left[\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx\right]_{n}}\left[\delta\phi\right]_n$$

The calculation is stopped when finally $\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx$ is, say, $\frac{1}{1000}$ of its value in the first calculation, or when inspection shows that ϕ_1 has reached a minimum value.

For those sets of parameters which do not yield a minimum ϕ_1 (see Section 4), the value of $\int_0^L \left(\sum_{i=1}^{n}\lambda_i \frac{\partial f_i}{\partial \alpha}\right)^2 dx$ in successive computations is progressively larger, as one might expect. Cases 4 and 6 in Table I are examples. $\delta\phi$ in these cases is set to be constant in successive computations.

Fig. 1 is a typical example showing the pattern of descent of ϕ_1 in arriving at its minimum value. The exit pressure is used as the abscissa to indicate the manner of convergence. Other quantities, such as temperature, can also be used.

In contrast to Fig. 1, a typical case of progressively descending ϕ_1, but with no apparent minimum ϕ_1 is shown in Fig. 2. It is of interest to note that when a minimum ϕ_1 exists, progressively ϕ_1 moves along a parabola-like path. When a minimum ϕ_1 does not exist, ϕ_1 moves along a hyperbola-like path.

Table I presents the results for $\sigma = T^{\omega}$, where $\omega = 10$ with the exception of case 6 in which $\omega = 2$. As shown in this table various values of the parameters involved are assigned. The results are summarized in the last 6 columns in the following order: inlet stagnation enthalpy ϕ_0, <u>optimum</u> exit stagnation enthalpy ϕ_1, exit stagnation enthalpy calculated in the first try $\overline{\phi}_1$ (in all cases except case 7 the first try is a constant velocity distribution, i.e, $\alpha_i = 0$), fraction of power extracted with optimum path $(\phi_0 - \phi_1) / \phi_0$, fraction of power extracted in the first try $(\phi_0 - \overline{\phi}_1) / \phi_0$, and the percentage gain of power extracted with optimum path as compared to the first try $(\overline{\phi}_1 - \phi_1) / (\phi_0 - \overline{\phi}_1)$.

The set of computations from case 1 to 4 have identical entrance conditions except the entrance Mach number. The results showed that ϕ_1 has a minimum only when $M_0^2 = 0.5$, 1.0, and 1.5. For the case of $M_0^2 = 2.0$ the decrease of ϕ_1 in successive computations is like that given in Fig. 2. The behavior of this type of descent will be discussed later. Velocity variation for optimum ϕ_1 of these runs are shown in Fig. 3. The local Mach number and channel cross-sectional area are also calculated for these optimum cases and are given in Fig. 4 and 5, respectively. Calculations using different size of steps in the iterations show that the final results are independent of the size of the steps taken in each of the successive computations.

Case 5 is like that of Case 1 except the value of γ is taken as 1.67 instead of 1.2. This change results in a large increase of extracted power. The velocity, local Mach number, and cross-sectional area for optimum condition are shown in Fig. 6.

In case 6, $\omega = 2$ was used. It was found that similar to case 4, there was a progressive descent of ϕ_1, but with no minimum.

Case 7 was assigned an $\alpha_o = 1$ in the first computation. As noted before, the slope of $\mathcal{U}$ at $x = 1$ will then always remain unity in all the successive computations. Velocity, local Mach number, and cross-sectional area for optimum conditions are shown in Figs. 3 through 5, respectively. Because of this intrinsic property of the first assigned value of α, the velocity variation is slightly different from that of case 1, which has an $\alpha_o = 0$. The resulting channel cross-sectional area and the optimum ϕ_1 for both cases, however, are exactly the same with any difference appearing only in the 4th or 5th significant figure.

Cases 8 through 10 were designed to find the most favorable entrance Mach number among all the optimum ϕ_1's for constant stagnation enthalpy at inlet. If Q_o is taken to be unity for $M_o^2 = 1$, then

$$\frac{Q_o}{(Q_o)_{M_o^2=1}} = Q_o = \frac{\left(T_o^{\omega-1/2} / M_o P_o\right)}{\left(T_o^{\omega-1/2} / M_o P_o\right)_{M_o^2=1}}$$

Using

$$T_o = \frac{T_{o,stag}}{1 + \frac{\gamma-1}{2} M_o^2}$$

we obtain
$$G_0 = \frac{\frac{1}{M_0}\left(1 + \frac{\gamma-1}{2} M_0^2\right)^{0.5+\frac{1}{\gamma-1}-\omega}}{\left(1 + \frac{\gamma-1}{2}\right)^{0.5+\frac{1}{\gamma-1}-\omega}}$$

If $\gamma = 1.2$, $\omega = 10$, then $G_o = \frac{1.537}{M_o} \left(1 + 0.1 M_o^2\right)^{4.5}$.

This variation of G_o with M_o was adopted in cases 8, 9, 10. ($M_o^2 = 0.5, 1.5, 2.0$, respectively. $M_o^2 = 1$ is already covered in case 2.)

Velocity, Mach number, and cross-sectional area of these cases are shown in Figs. 7, 8, and 9 respectively. From Fig. 10, it is seen that the most favorable entrance Mach number is near unity.

Similar calculations were performed in case 11 for $\gamma = 1.67$. Again, G_o was chosen to be unity for $M_o^2 = 1.0$. The results of case 11 are plotted in Fig. 6.

It was noted that for conductivity taking the form of $\sigma = T^{\omega}$, cases 4 and 6 do not have a minimum ϕ_1. ϕ_1 in these cases is progressively smaller in successive computations, and p_1 and T_1 progressively approach zero. This behavior can be seen from Eqs. (16b') and (16c'). Pressure and temperature will approach zero when $\frac{dp}{dx}$ and $\frac{dT}{dx}$ become excessively negative throughout the channel. Now, when the terms within the parenthesis in these equations assume a negative value, then p and T will always be positive and cannot approach zero. When these terms assume a positive value, then $\frac{dp}{dx}$ and $\frac{dT}{dx}$ become excessively negative if M_o^2 or γ, or both take a large value, and hence p_1 and T_1 approaches zero at the exit. The first term inside the parenthesis in both equations is always positive. In the case of a nearly-constant-velocity channel, say, this term is excessively large when ω is small. The above is a heuristic explanation of case 4, where M_o^2 becomes excessively large, and of case 6, where ω is not sufficiently large.

II Conductivity varies as $\dfrac{T^\omega}{\sqrt{p}}$

When conductivity is taken as a function of both temperature and pressure, and is of the form $\sigma = \dfrac{T^{10}}{\sqrt{p}}$

Eqs. (16a), (16b), and (16c) become:

$$\frac{du}{dx} = \alpha \tag{16a''}$$

$$\frac{dp}{dx} = -\gamma M_o^2 \left(C u \frac{T^{10}}{p^{1/2}} + p \frac{u\alpha}{T} \right) \tag{16b''}$$

$$\frac{dT}{dx} = -(\gamma-1) M_o^2 \left(K C u \frac{T^{10}}{p^{3/2}} + u\alpha \right) \tag{16c''}$$

Eqs. (21a), (21b), and (21c) become:

$$\frac{d\lambda_u'}{dx} = \left(C \frac{T^{10}}{p^{1/2}} + \frac{p\alpha}{T} \right) \lambda_p' + \left(K C \frac{T^{11}}{p^{3/2}} + \alpha \right) \lambda_T \tag{21a''}$$

$$\frac{d\lambda_p'}{dx} = \gamma M_o^2 \left\{ \left(-C \frac{u T^{10}}{2 p^{3/2}} + \frac{\alpha u}{T} \right) \lambda_p' - \frac{3}{2} K C \frac{u T^{11}}{p^{5/2}} \lambda_T \right\} \tag{21b''}$$

$$\frac{d\lambda_T}{dx} = (\gamma-1) M_o^2 \left\{ \left(10 C \frac{u T^9}{p^{1/2}} - \frac{\alpha u p}{T^2} \right) \lambda_p' + 11 K C \frac{u T^{10}}{p^{3/2}} \lambda_T \right\} \tag{21c''}$$

Cases 12 to 15 in Table II list the given conditions and results in the series of computations carried out with this form of conductivity. None of this series of computation gives a minimum ϕ_1. ϕ_1 in successive computations descends like that given in Fig. 2. It appears that the heuristic argument outlined before for cases 4 and 6 is equally applicable to explain the result. Fig. 11 is a typical example of the successive steps assumed by velocity, pressure, temperature, and cross-sectional area in the iteration.

Since zero pressure and temperature at the exit implies infinite cross-sectional area, one is naturally interested only in exit pressure and temperature which give an exit cross-sectional area of practical value. For this purpose, an interpolated ϕ_1 is obtained for a fixed exit-entrance area ratio. ϕ_1 thus obtained is tabulated in Table II for $\frac{A_1}{A_o} = 10$, and 20. Similar results for cases 4 and 6 are also given in Table II. The blank in case 15 for $\frac{A_1}{A_o} = 10$ is due to the fact that the constant velocity ($\alpha = 0$) calculation already yields an $\frac{A_1}{A_o} = 13.34$. In all cases, the advantage over constant velocity distribution is large.

Acknowledgements

The authors are pleased to express their deep gratitude to Dr. George W. Sutton for suggesting this problem as well as many interesting discussions. They are also indebted to Professor Arthur E. Bryson of Harvard University for calling their attention to Reference 6.

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Table I Results of Computation for $\sigma = T^{\omega}$ $(K=\frac{1}{2})$

case no.	α_o*	M_o^2	ω	γ	Q_o	ϕ_o	Optimum ϕ_1	$\bar{\phi}_1$ (from $\alpha=\alpha_o$)	$\frac{\phi_o-\phi_1}{\phi_o}\%$	$\frac{\phi_o-\bar{\phi}_1}{\phi_o}\%$	$\frac{\phi_1-\bar{\phi}_1}{\phi_o-\phi_1}\%$
1 or 1A	0	0.5	10	1.2	1.0	1.05	1.02307	1.02467	2.56	2.41	6.32
2	0	1.0	10	1.2	1.0	1.10	1.04851	1.04853	4.68	4.68	0.03
3	0	1.5	10	1.2	1.0	1.15	1.06894	1.07129	7.04	6.84	2.97
4	0	2.0	10	1.2	1.0	1.20		1.09257			
5	0	0.5	10	1.67	1.0	1.1675	1.07905	1.09930	7.58	5.84	30
6	0	0.5	2	1.2	1.0	1.05		1.02181			
7 or 1B	1	0.5	10	1.2	1.0	1.05	1.02333	1.02567	2.48	2.31	9.6
8	0	0.5	10	1.2	1.744	1.05	1.00178	1.00532	4.59	4.25	7.9
9	0	1.5	10	1.2	0.668	1.15	1.09689	1.09838	4.62	4.49	2.9
10	0	2.0	10	1.2	0.478	1.20	1.14567	1.15087	4.53	4.09	10.6
11	0	0.5	10	1.67	4.120	1.1675	0.932160	0.980668	20.1	16.0	26.5

* α_o = assigned $\alpha(x)$ in the first of the successive computations

Table II Results of Computation for Cases Without Minimum ϕ_1
($K=\frac{1}{2}$)

case no.	α_0 *	M_0^2	σ	ω	γ	Q_0	ϕ_0	lowest ϕ_1 if: $\frac{A_1}{A_0}=10$	lowest ϕ_1 if: $\frac{A_1}{A_0}=20$	$\bar{\Phi}_1$ (from $\alpha=\alpha_0$)	$\frac{\phi_0-\phi_1}{\phi_0}\%$ if: $\frac{A_1}{A_0}=10$	$\frac{\phi_0-\phi_1}{\phi_0}\%$ if: $\frac{A_1}{A_0}=20$	$\frac{\phi_0-\bar{\Phi}_1}{\phi_0}\%$	$\frac{\bar{\Phi}_1-\phi_1}{\phi_0-\bar{\Phi}_1}\%$ if: $\frac{A_1}{A_0}=10$	$\frac{\bar{\Phi}_1-\phi_1}{\phi_0-\bar{\Phi}_1}\%$ if: $\frac{A_1}{A_0}=20$
12	0	0.5	$T^{\omega}/\sqrt{p}$	10	1.2	1	1.05	1.004	0.993	1.0226	4.4	5.4	2.6	64	108
13	0	1.0	$T^{\omega}/\sqrt{p}$	10	1.2	1	1.10	1.012	1.003	1.0384	8.0	8.8	5.6	43	56
14	0	1.5	$T^{\omega}/\sqrt{p}$	10	1.2	1	1.15	1.096	0.999	1.0412	11.7	13.1	9.5	23	38
15	0	2.0	$T^{\omega}/\sqrt{p}$	10	1.2	1	1.20		0.992	1.0053		17.3	16.2		7
4	0	2.0	T^{ω}	10	1.2	1	1.20	1.037	1.0125	1.0926	13.6	15.6	9.0	40	63
6	0	0.5	T^{ω}	2	1.2	1	1.05	0.929	0.901	1.0218	11.5	14.2	2.7	330	430

* α_0 = assigned $\alpha(x)$ in the first of successive computations

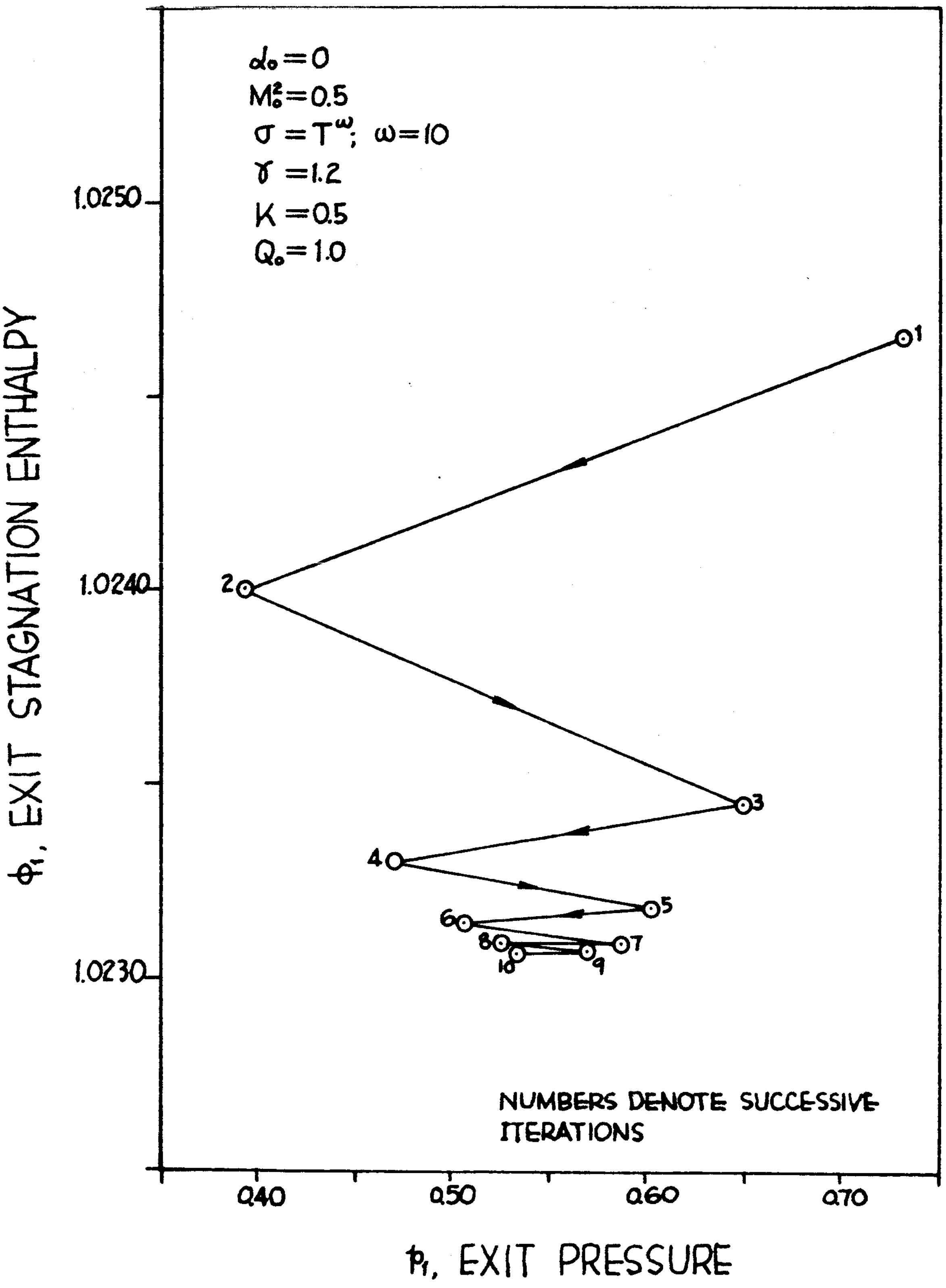

FIG. I ITERATION PATTERN FOR A CASE WITH MINIMUM Φ_1

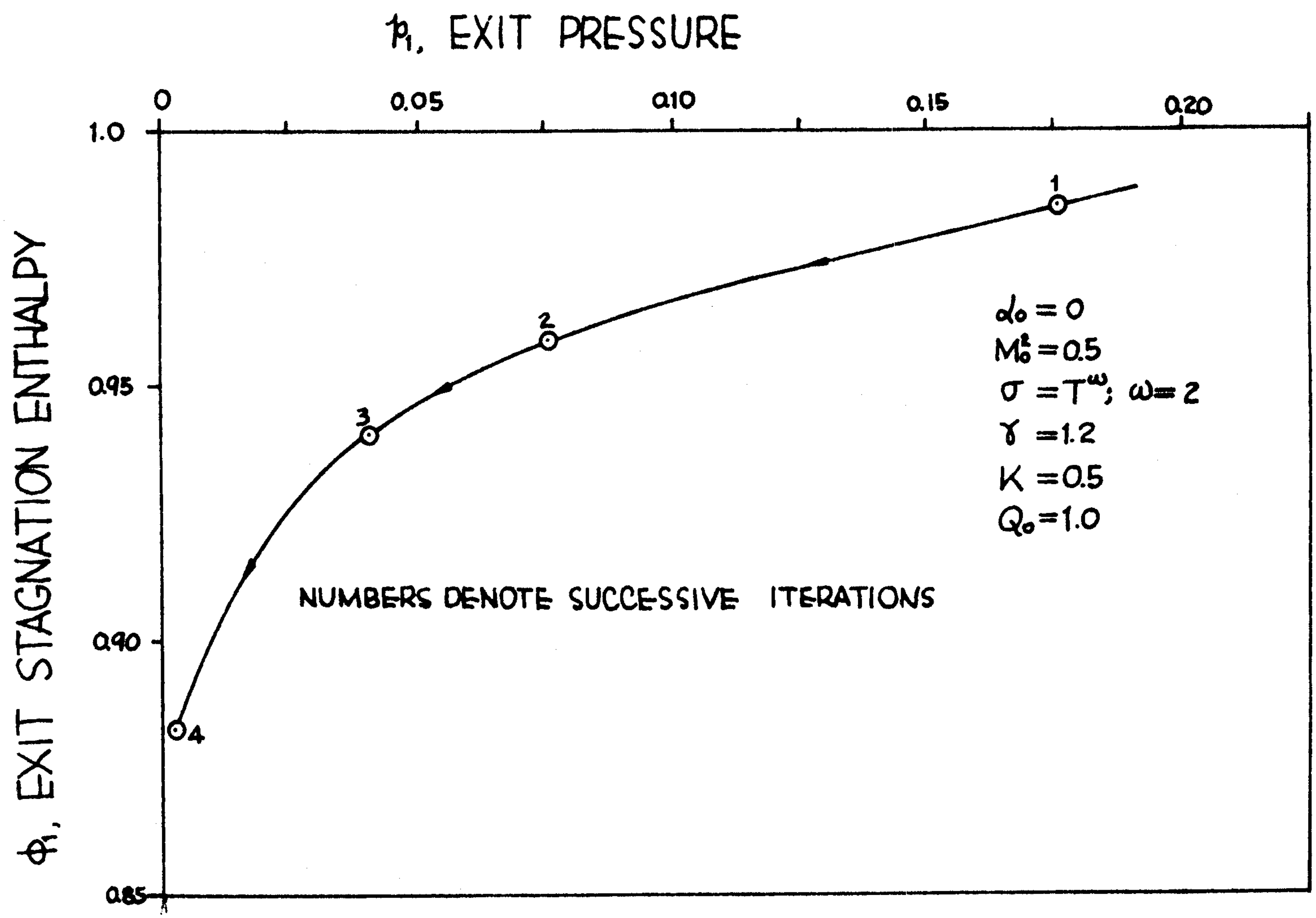

FIG. 2 ITERATION PATTERN FOR A CASE WITHOUT MINIMUM Φ_1

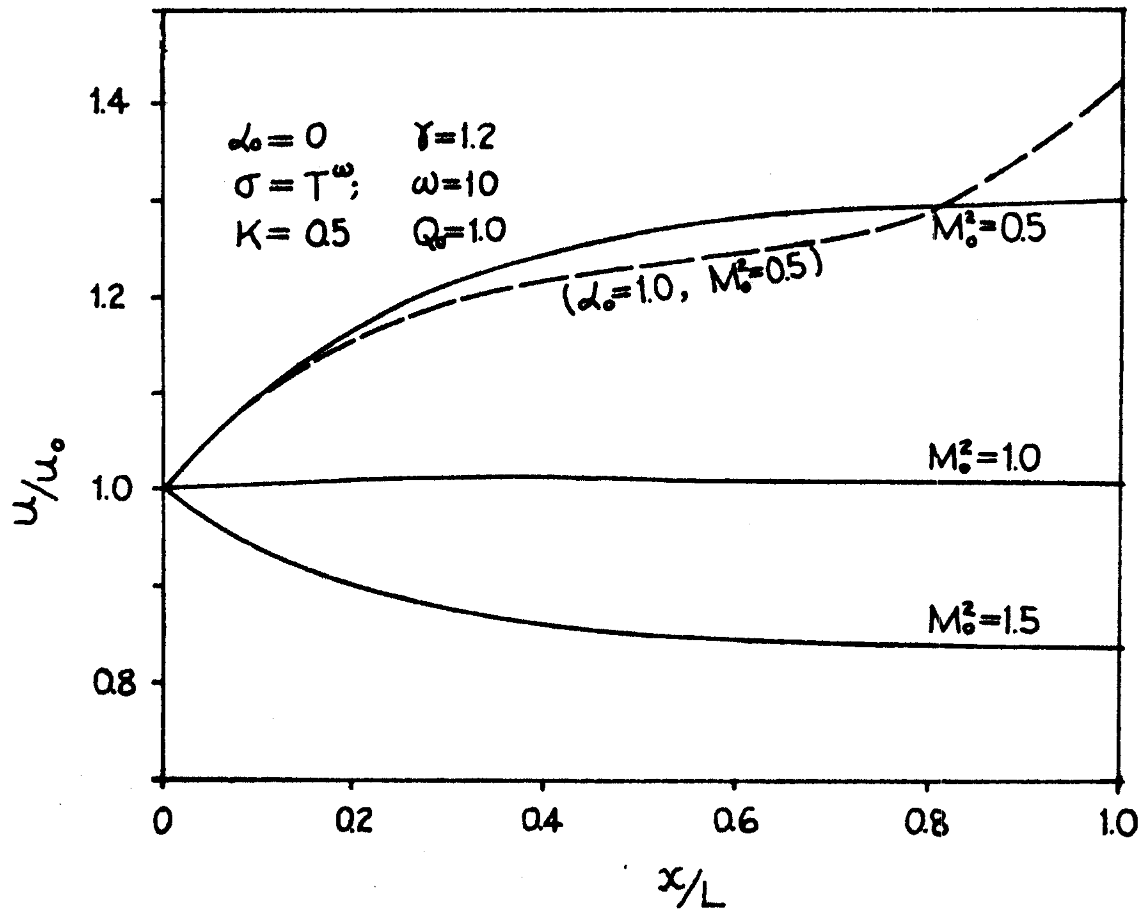

FIG. 3 VELOCITY DISTRIBUTION FOR DIFFERENT INLET MACH NUMBERS WITH $Q_o = 1$

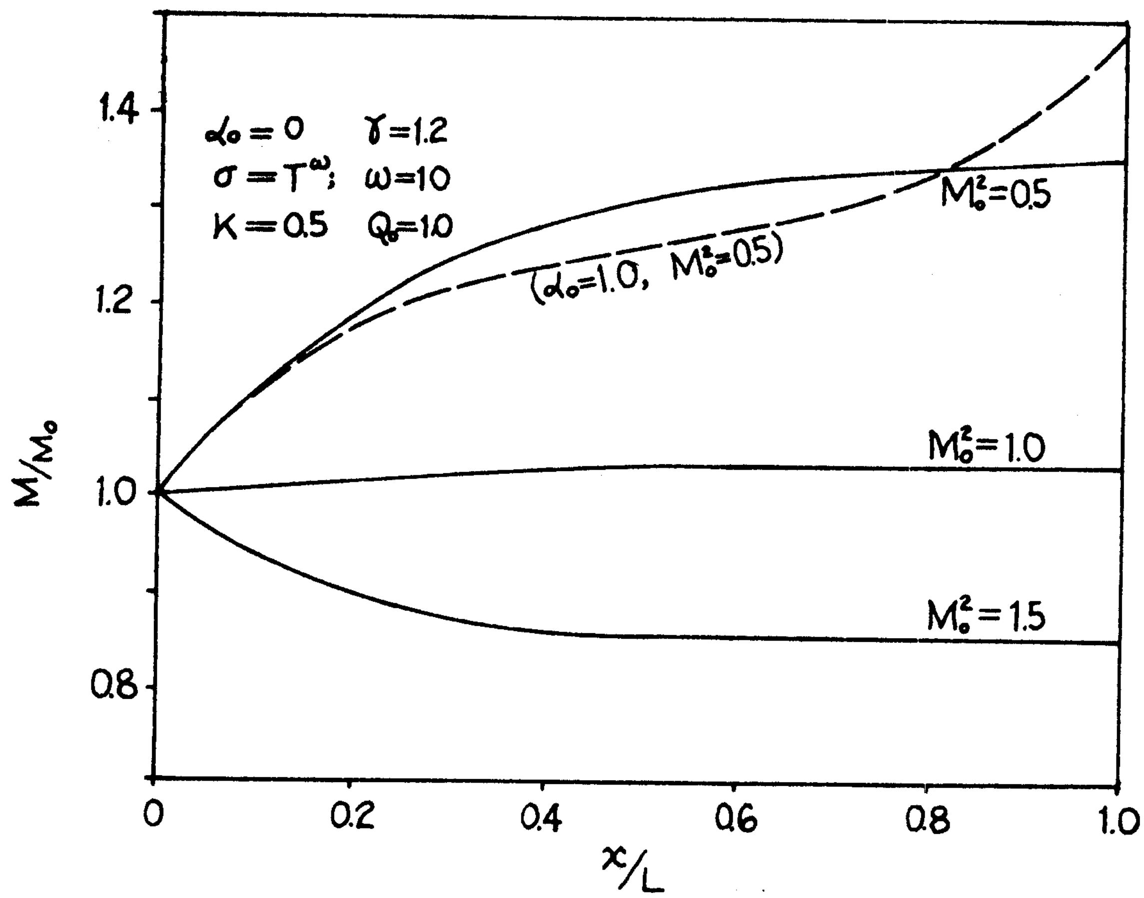

FIG. 4 MACH NUMBER DISTRIBUTION FOR DIFFERENT INLET MACH NUMBERS WITH $Q_o = 1$

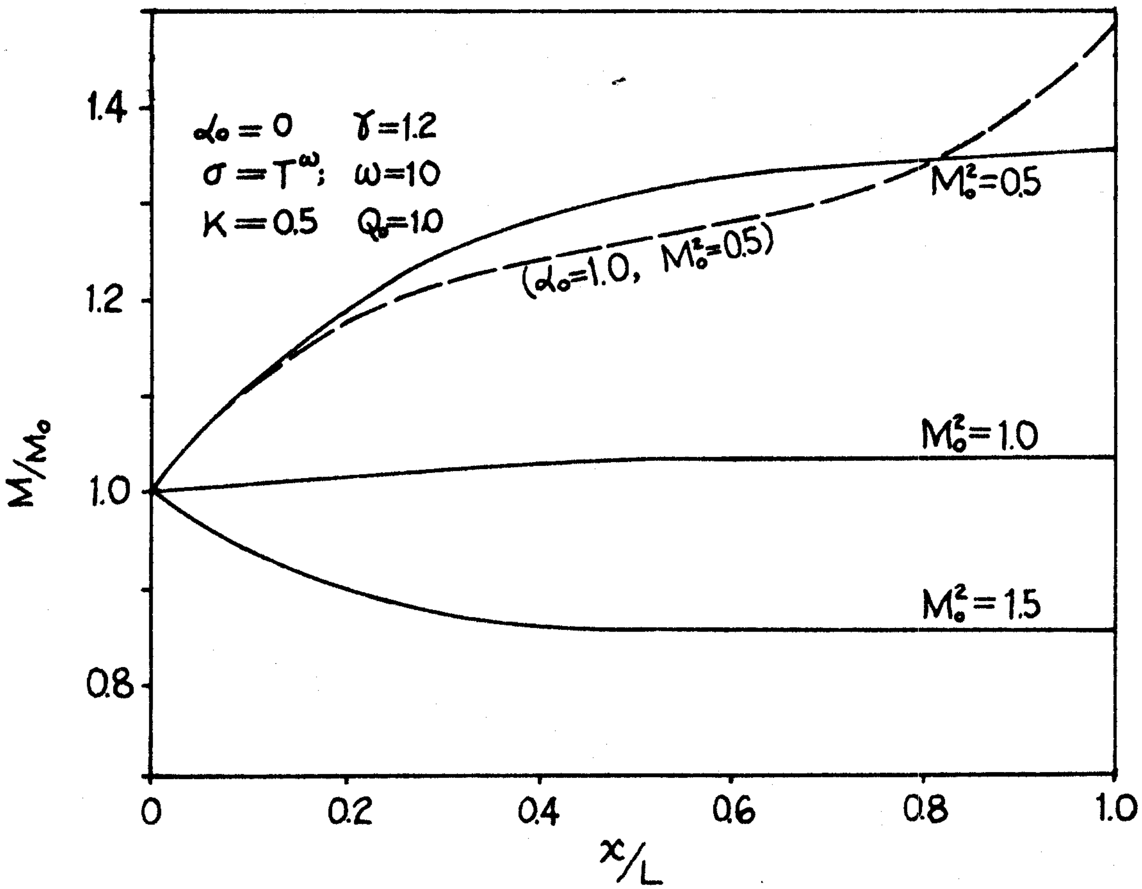

FIG. 4 MACH NUMBER DISTRIBUTION FOR DIFFERENT INLET MACH NUMBERS WITH $Q_o = 1$

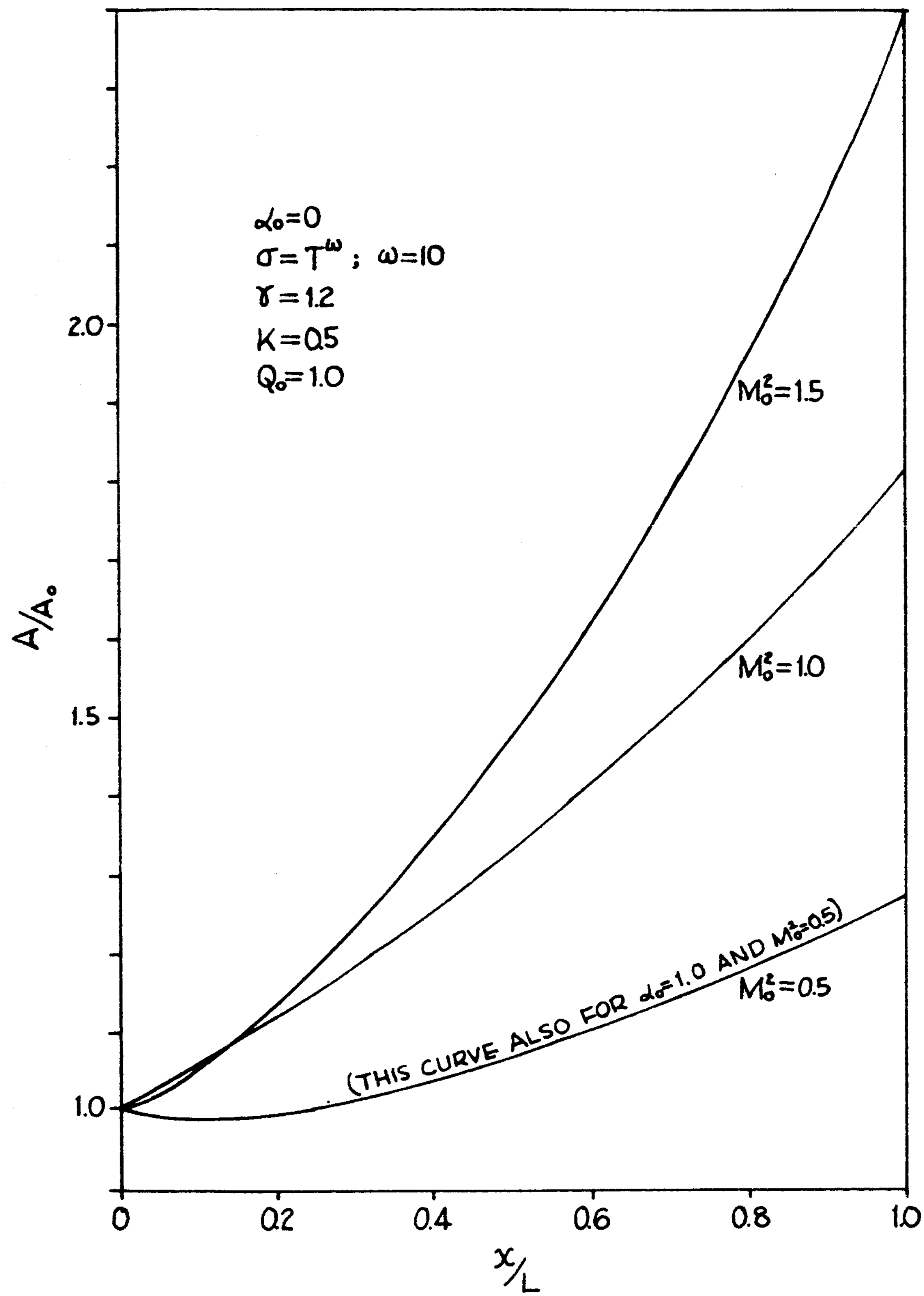

FIG. 5 CROSS-SECTIONAL AREA DISTRIBUTION FOR DIFFERENT INLET MACH NUMBERS WITH $Q_o = 1$

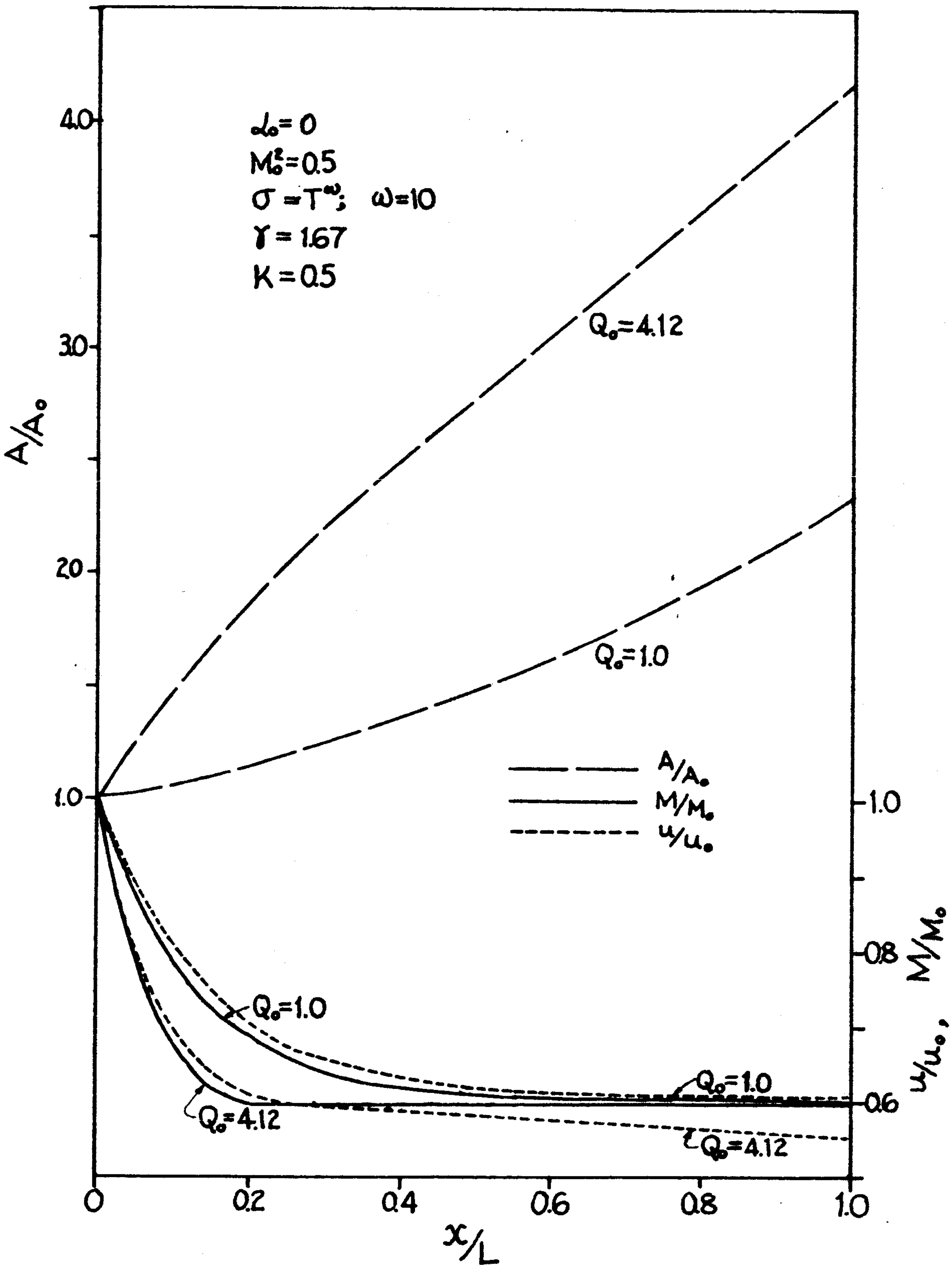

FIG. 6 VELOCITY, MACH NUMBER, AND CROSS-SECTIONAL AREA DISTRIBUTIONS FOR $\gamma = 1.67$

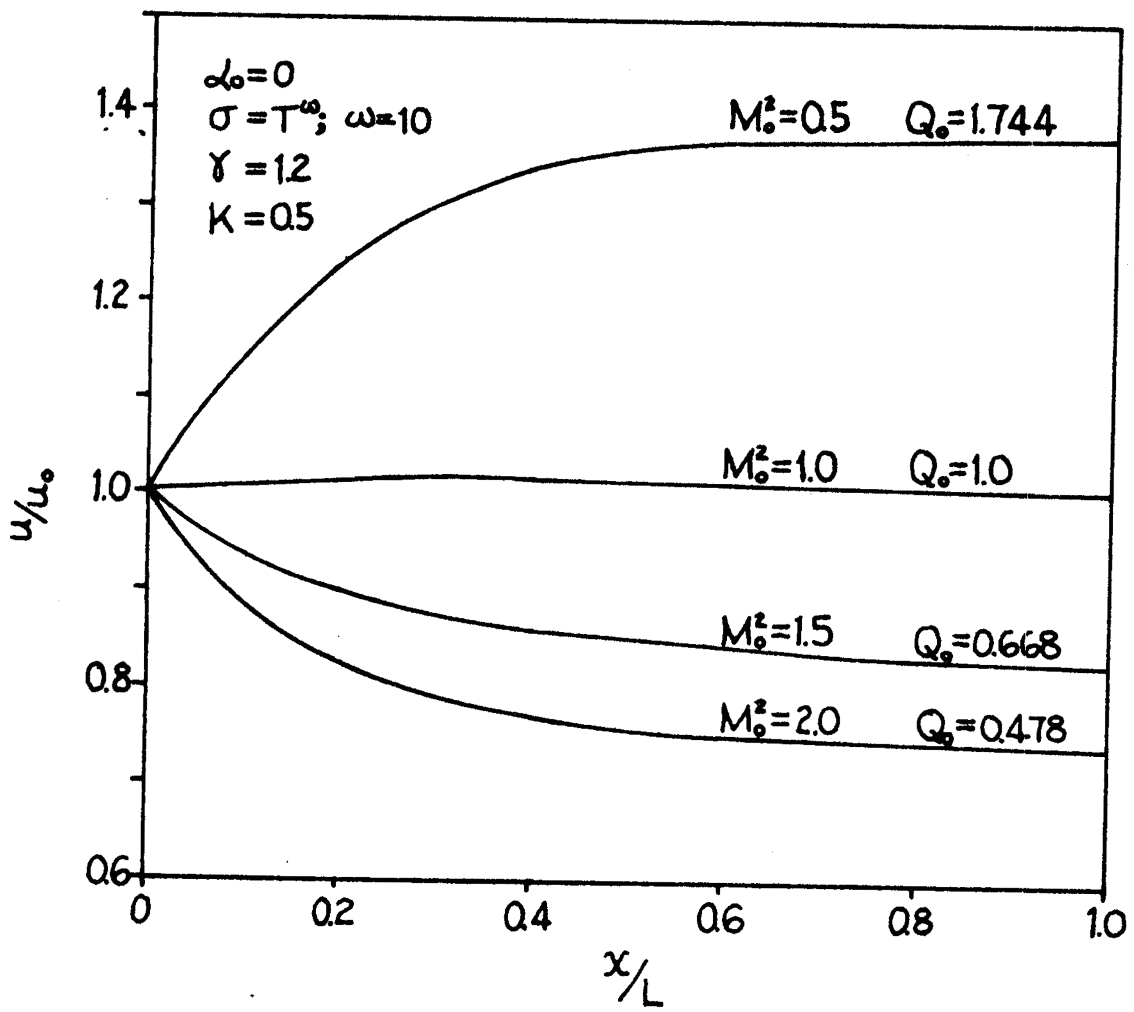

FIG. 7 VELOCITY DISTRIBUTION FOR DIFFERENT INLET MACH NUMBERS WITH CONSTANT INLET STAGNATION ENTHALPY

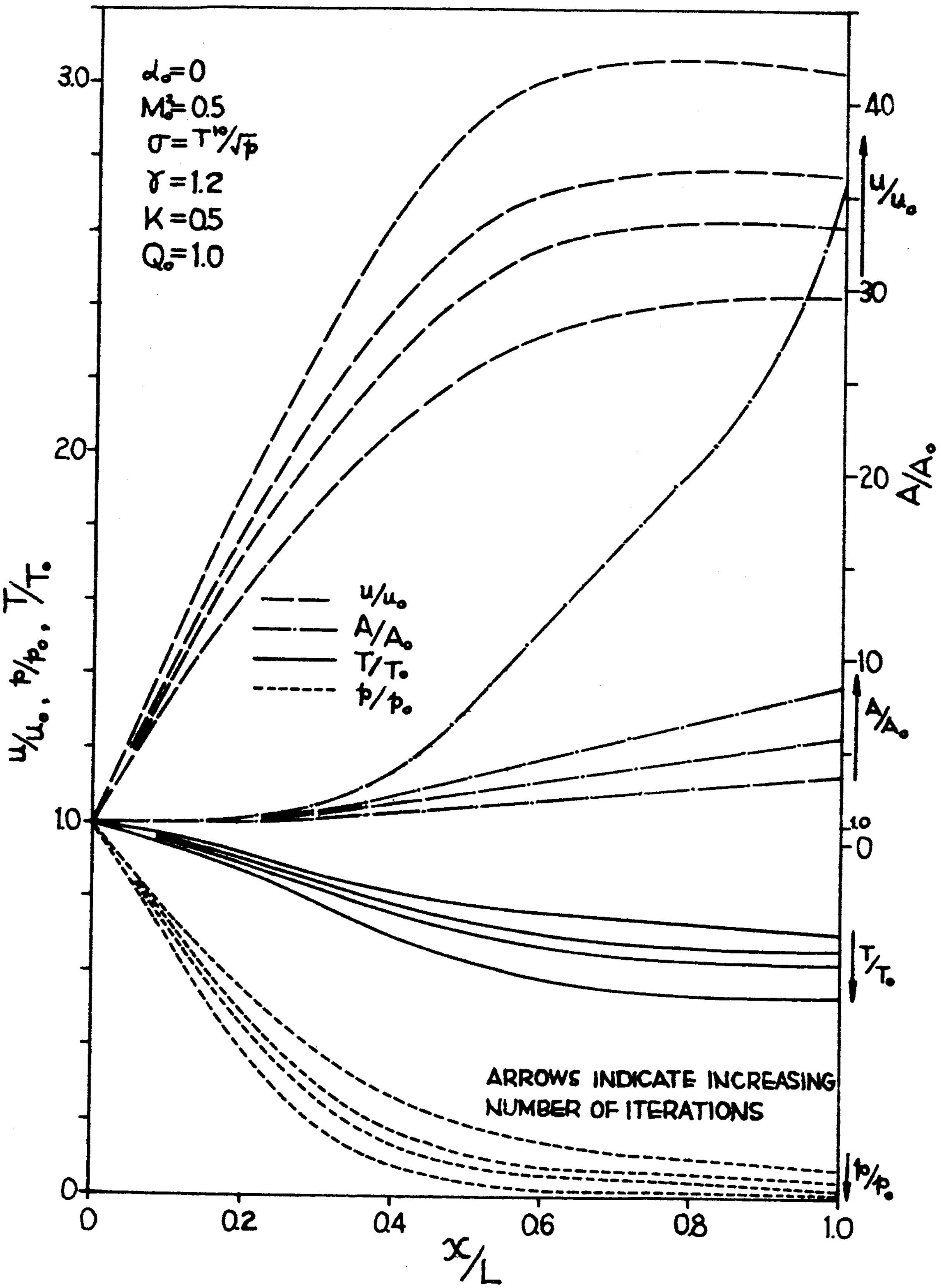

FIG. 11 VARIATION OF VELOCITY, PRESSURE, TEMPERATURE, AND CROSS-SECTIONAL AREA FOR A CASE WITHOUT MINIMUM